Centrifugal Drum Filtration: II. A Compression Rheology Model of Cake Draining

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A compression rheology model for the final capillary compression and draining stages of the batch centrifugal filtration of a compressible cake is presented. The initial stage of cake formation and drainage of the supernatant fluid have been treated in a companion paper up to the point t_c, where the supernatant first contacts the top of the cake. The compressional rheology model uses the compressive yield stress $p_y(\phi)$, the hydrodynamic resistance $R(\phi)$, and the maximum capillary stress $P_{cap}^{max}(\phi)$. Analysis of these final stages indicates that the system can remain fully saturated if the solids network can sustain the capillary stress and, if not, will partially unsaturate. The controlling parameters are elucidated and the state of the system for $t_c < t < \infty$ is illustrated graphically for the previously treated model system. © 2005 American Institute of Chemical Engineers *AIChE J,* 52: 557–564, 2006

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Introduction

The draining of liquid from a cake of solids during centrifugal filtration was previously analyzed by Wakeman and coworkers.[1-3] We present an alternative method for the solution of the problem of centrifugally deliquoring a compressible cake using a compressional rheology model coupled with the thermodynamic analysis of the advance of a liquid interface through a porous solid phase. The compression rheology model presented previously[4] determines the structure at the onset of the capillary compression stage.

We model a drum of radius r_m, spinning at constant speed ω, having been instantaneously loaded with a volume per length v_0, of a suspension with solids volume fraction ϕ_0 ($<\phi_g$). Under these conditions, centrifugal drum filtration becomes a two-dimensional problem in the radial coordinate r and time t once turbulence and edge effects have been neglected for a uniformly loaded drum. At time t, $v(t)$ is the expressed filtrate volume per length of the drum and the solids concentration profile is $\phi(r, t)$. The force balances in the solid and liquid phases under these conditions are

$$\frac{\partial p_s}{\partial r} = \Delta\rho\phi\omega^2 r - R(\phi)\left(\phi \boldsymbol{u} - \frac{\phi\dot{v}}{2\pi r}\right) \tag{1}$$

and

$$\frac{\partial p_l}{\partial r} = \rho_l\omega^2 r + R(\phi)\left(\phi \boldsymbol{u} - \frac{\phi\dot{v}}{2\pi r}\right) \tag{2}$$

where $\Delta\rho = \rho_s - \rho_l$, $\boldsymbol{u}$ is the solids velocity, and $R(\phi)$ is the hydrodynamic resistance.[5] We use the compressive rheology closure introduced by Buscall and White[6]

$$p_s(\boldsymbol{r}, t) = p_y[\phi(\boldsymbol{r}, t)] \tag{3}$$

in consolidating regions of the cake. If the solids pressure is less than the yield stress $\{p_s(\boldsymbol{r}, t) < p_y[\phi(\boldsymbol{r}, t)]\}$ the solid network will not compress.

The solids flux in a region of consolidating networked solids, from Eqs. 1 and 3 is

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$$\phi \boldsymbol{u} = \Delta\rho\omega^2 r \frac{\phi}{R(\phi)} + \frac{\phi \dot{v}}{2\pi r} - D(\phi)\frac{\partial \phi}{\partial r} \tag{4}$$

where $D(\phi)$ is the solids diffusion coefficient.[7]

$$D(\phi) = \frac{\partial p_y(\phi)}{\partial \phi} \Big/ R(\phi) \tag{5}$$

As discussed earlier in the companion article,[4] $p_s(r_m, t)$ peaks at t_I and an immobilized zone where $p_s(r, t) < p_y[\phi(r, t)]$ moves out from the membrane. The boundary between the immobilized and compressing cake is denoted $r_I(t)$. The fluid–air interface is at $r_f(t)$, where

$$2\pi \int_0^{r_f} (1 - \phi) r dr = (1 - \phi_0) v_0 - v(t) \tag{6}$$

The inner cake surface is situated at $r_c(t)$. The previous analysis[4] ended at the point t_c where $r_c(t_c) = r_f(t_c)$ and the fluid–air interface first contacts the cake surface. We showed that by this time the entire cake had immobilized $[r_I(t_c) = r_c(t_c) = r_f(t_c)]$. The immobilized solids profile at this point is $\phi(r, t_c)$ and is calculated using the algorithm previously outlined.[4] The total pressure, $p_t = p_s + p_l$, can be evaluated at the membrane as

$$p_t(r_m, t) = p_a + \frac{\rho_l \omega^2}{2\pi}\left(1 + \frac{\Delta\rho}{\rho_l}\phi_0\right) v_0 - \frac{\rho_l\omega^2}{2\pi} v \tag{7}$$

and

$$p_l(r_m, t) = \frac{R_{mem}}{2\pi r_m}\dot{v}(t) + p_a \tag{8}$$

provided $r_f \leq r_c$, that is, the cake remains fully saturated.

Cake Compression

At t_c, $p_l(r_c, t_c) = p_a$ and $p_s(r_c, t_c) = 0$, corresponding to a solids concentration $\phi(r_c, t_c) = \phi_g$. Because $\dot{V} > 0$, the subsequent removal of fluid will cause the appearance of liquid–air menisci at the cake surface. The fluid pressure at the cake surface becomes

$$p_l(r_c, t) = p_a - p_{cap}(t) \tag{9}$$

where the capillary (Laplace) pressure is

$$p_{cap}(t) = \frac{2\gamma_{LV}}{r_{eff}(t)} \tag{10}$$

where $r_{eff}(t)$ is the average radius of curvature of those menisci at time t ($>t_c$) and γ_{LV} is the liquid–vapor surface tension[8] (see Figure 1).

The total pressure at the interface remains at p_a. Consequently

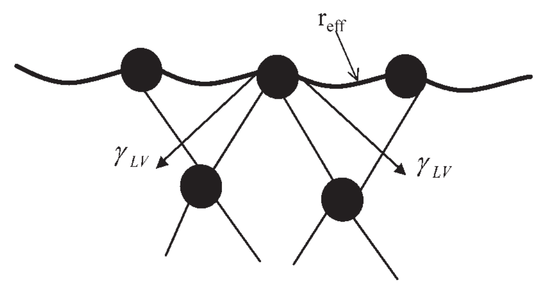

Figure 1. Menisci form between solid particles as liquid drains from the cake surface.

Surface tension generates a compressive pressure on the particles of the solids network at the surface and a corresponding pressure in the liquid.

$$p_s(r_c, t) = p_{cap}(t) \geq 0 \tag{11}$$

and the surface solids concentration must increase accordingly:

$$p_y[\phi(r_c, t)] = p_{cap}(t) \tag{12}$$

Fluid drainage causes $r_{eff}(t)$ to decrease and $\phi(r_c, t)$ to increase. Thus the action of the liquid–air interface is to recommence compression of the cake ($u \neq 0$) near the cake surface. The underlying cake remains immobile until $p_s(r, t) = p_y[\phi(r, t)]$. The boundaries $r_I(t) > r_c(t) = r_f(t)$ all move outward in this capillary compression stage.

Scaled equations

To describe this situation, we use the compression rheology equations outlined above in scaled form:

$$P = \frac{2p}{\rho_l \omega^2 r_m^2} \qquad T = \frac{t}{t^*} \qquad \lambda = \frac{R(\phi)}{R^*} \qquad V = \frac{v}{\pi r_m^2}$$
$$Z = \frac{r_m^2 - r^2}{r_m^2} \qquad \Delta(\phi) = D(\phi)\frac{2R^*}{\rho_l\omega^2 r_m^2} \qquad \psi = \frac{2t^*}{r_m^2} r\phi u \tag{13}$$

where t^* and R^* are chosen so that

$$t^* = \frac{R^*}{2\rho_l\omega^2} \tag{14}$$

The resistance scaling[4] is

$$R^* = \frac{\rho_l \omega^2 r_m^2}{2D[\phi(0, T_I)]} \tag{15}$$

Conservation of solids yields[4]

$$\frac{\partial \phi}{\partial T} = \frac{\partial \psi}{\partial Z} \tag{16}$$

The scaled solids flux in the compressing region is

$$\psi(Z, T) = \phi\dot{V} + (1 - Z)\left[\frac{\Delta\rho}{\rho_l}\frac{\phi}{\lambda} + \Delta(\phi)\frac{\partial\phi}{\partial Z}\right] \quad (17)$$

We define

$$I_1(Z) = \int_0^Z \phi dZ \quad (18)$$

$$I_2(Z) = \int_0^Z \frac{\phi\lambda}{1 - Z} dZ \quad (19)$$

$$I_3(Z) = \int_{Z_I}^Z \frac{\psi\lambda}{1 - Z} dZ \quad (20)$$

The scaled solids pressure at the immobilization boundary, $Z_I(T)$, is[4]

$$P_s(Z_I, T) = P_y(\phi_I) = L\left(1 + \frac{\Delta\rho}{\rho_l}\phi_0\right) - V - \dot{V}[\mu + I_2(Z_I)] - \frac{\Delta\rho}{\rho_l} I_1(Z_I) \quad (21)$$

where

$$\mu = \frac{2R_{mem}}{R^* r_m} \quad (22)$$

The solids velocity at $r_c(t)$ is

$$u(r_c, t) = \frac{\dot{v}}{2\pi r_c} \quad (23)$$

because the top of the cake is moving with the fluid during the capillary compression stage. Thus

$$Z_f(T) = Z_c(T) = L - V(T) \quad (24)$$

from Eqs. 6 and 23 yields

$$\psi(Z_c, T) = \phi(Z_c, T)\dot{V}(T) \quad (25)$$

By integrating the liquid pressure Eq. 2 from the membrane to $r_f(t)$ using Eqs. 8 and 9 we obtain, in scaled form,

$$\dot{V}(T) = \frac{Z_f(T) - P_{cap}(T) + I_3(Z_f)}{\mu + I_2(Z_f)} \qquad (T > T_c) \quad (26)$$

The control parameters of the system are

$$\mathsf{M} = \frac{R_{mem}}{R_0} \qquad \mathsf{P}^* = \frac{\rho_l\omega^2 r_m^2}{2p_0} \quad (27)$$

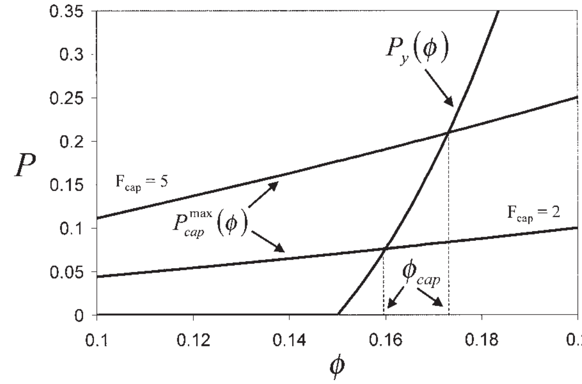

Figure 2. Compressive yield stress and maximum capillary stress (F_{cap} = 2, 5, P* = 5) as functions of solids concentration.

The intersection is ϕ_{cap}. Concentrations $> \phi_{cap}$ do not generate sufficient capillary stress to cause compression.

Unsaturation

The capillary pressure at the cake surface cannot increase indefinitely. White[9] has shown by a thermodynamic argument that the maximum capillary pressure that can be exerted in a rigid medium of volume fraction ϕ is

$$p_{cap}^{max} = \gamma_{LV}\cos(\Theta_C)\rho_s\bar{A}_s \frac{\phi}{1 - \phi} \quad (28)$$

where Θ_C is the solid–liquid contact angle and $\bar{A}_s$ is the specific surface area. Once $p_{cap} = p_{cap}^{max}$, the liquid phase moves inside the medium. When the solids concentration at the cake surface reaches the volume fraction at which the local solids have become sufficiently strong to resist further compression, the draining liquid has no alternative but to move inside the cake surface, that is, the cake begins to unsaturate. Thus there exists a critical volume fraction $\phi(Z_c, T) = \phi_{cap}$ at which

$$p_{cap}^{max}(\phi_{cap}) = P_y(\phi_{cap}) \quad (29)$$

(see Figure 2). The suspension property that will determine the final state of the system is

$$F_{cap} = \frac{\gamma_{LV}\cos(\Theta_C)\rho_s\bar{A}_s}{p_0} \quad (30)$$

Note that $F_{cap} = 2$ for 0.7-micron monodisperse spheres at a water–air interface with $p_0 = 1$ atmosphere and $\Theta_C = 0°$, and that $F_{cap} = 0.014$ for 100-micron spheres and $F_{cap} = 50$ for 28-nm spheres under the same conditions. Thus

$$p_{cap}^{max}(\phi) = \frac{F_{cap}}{\mathsf{P}^*}\frac{\phi}{1 - \phi} \quad (31)$$

and Eq. 29 becomes

$$P_y(\phi_{cap}) = \frac{F_{cap}}{\mathsf{P}^*} \frac{\phi_{cap}}{1 - \phi_{cap}} \quad (32)$$

If the cake surface concentration reaches this critical value for $\dot{V} > 0$, unsaturation of the cake will begin. We designate the time at which $\phi(Z_c, T) = \phi_{cap}$ as T_u. The subsequent drainage ($T > T_u$) of the unsaturated cake is discussed below.

If F_{cap} is sufficiently large then the compression of the cake will bring $\dot{V}$ to zero before $\phi(Z_c, T) = \phi_{cap}$, and thus no time T_u exists and filtration will cease (at $T = \infty$) with $\phi(Z_c, \infty) \leq \phi_{cap}$. We test for this possible final state by searching for a solids concentration profile that satisfies solids conservation, cake saturation, and $\phi(Z_c, \infty) \leq \phi_{cap}$ when liquid and solid flow have ceased. This zero flux limiting solids concentration profile, $\Phi(Z) = \phi(Z, \infty)$, in the consolidation regime obeys

$$\frac{\partial \Phi}{\partial Z} = -\frac{\Delta\rho}{\rho_l} \frac{\Phi}{\Delta(\Phi)\lambda(\Phi)} \quad (33)$$

from Eq. 17. We solve Eq. 33 from Z_I^∞ to Z_c^∞ subject to

$$P_y(\Phi_I) = L\left(1 + \frac{\Delta\rho}{\rho_l}\phi_0\right) - V^\infty - \frac{\Delta\rho}{\rho_l} I_1(Z_I^\infty) \quad (34)$$

$$I_1(Z_c^\infty) = L\phi_0 \quad (35)$$

$$L - V^\infty = Z_c^\infty \quad (36)$$

Note that Φ_I may lie on the immobilization profile, $\phi(Z, T_c)$, or the entire cake may have been mobilized during compression and $Z_I^\infty = 0$ with $\Phi_I > \phi_I(0, T_I)$. The initial test is started at $Z_I^\infty = 0$ and $\Phi_I = \phi_I(0,T_I)$. If $Z_c^\infty > L - V^\infty$, the fully saturated profile starts at the membrane with $\Phi(0) > \phi(0, T_c)$; otherwise, $0 < Z_I^\infty \leq Z_I(T_c)$ and Φ_I lies on the immobilization profile, $\phi(Z, T_c)$. For each test we choose Z_I^∞ or $\Phi(0)$ as appropriate, then use Eq. 34 to determine V^∞ and solve Eq. 33 from Z_I^∞ to Z_c^∞ where Eq. 35 is satisfied. If Eq. 36 is also satisfied, the zero flux limiting solids concentration profile has been determined; otherwise, we select a new Z_I^∞ or $\Phi(0)$ and repeat the calculation. Once the limiting profile is determined we examine the value of $\Phi(Z_c^\infty)$.

If $\Phi(Z_c^\infty) > \phi_{cap}$ the cake will (partially) unsaturated because the capillary pressure required to stop the filtration while the cake remains saturated cannot be attained. If $\Phi(Z_c^\infty) \leq \phi_{cap}$ the cake will remain saturated. If the cake remains saturated for a given value of F_{cap}, then it will remain so for all higher values of F_{cap} because the capillary stress on the liquid will always be sufficient to stop expression for larger values of F_{cap}. Furthermore, the infinite time concentration profile will remain the same for greater values of F_{cap} because the stresses experienced by the cake will not change.

Thus the final state of centrifugal cake drainage can be plotted on a "phase diagram " (Figure 3), where regions of (F_{cap}, P^*) space, which yield fully and partially saturated final states, are delimited by a curve that is only weakly sensitive to the membrane resistance, as expected.

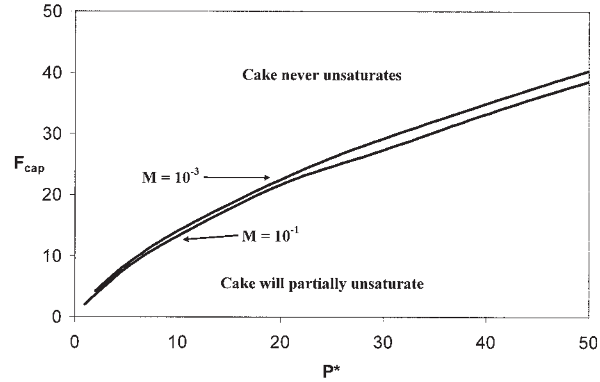

Figure 3. Cake final saturation as function of F_{cap}, P*, and M for the model system described in Table 1.

Note that whether the cake unsaturates is relatively insensitive to membrane resistance.

Cake compression algorithm

Once the final saturation state of the filter cake has been determined, we can evaluate the solutions for times up to the point where filtration stops ($T = \infty$) or the cake begins to unsaturate ($T = T_u$). It is convenient to select a value of $Z_I^{(k)}$ [$< Z_I^{(k-1)}$]. Then $\phi_I^{(k)}$ is determined from the $\phi[Z, T^{(k-1)}]$ profile. If $Z_I[T^{(k)}] = 0$ is reached, then $\phi_I^{(k)} > \phi[0, T^{(k-1)}]$. Using an estimate for $\dot{V}^{(k)}$, $V^{(k)}$ is determined from Eq. 21. The value of $T^{(k)}$ and $Z_c^{(k)}$ are calculated from

$$\Delta T^{(k)} = 2\Delta V^{(k)}/[\dot{V}^{(k)} + \dot{V}^{(k-1)}] + \cdots \quad (37)$$

and

$$\Delta Z_c^{(k)} = \frac{\Delta T^{(k)}}{2}[\dot{Z}_c^{(k)} + \dot{Z}_c^{(k-1)}] + \cdots \quad (38)$$

accurate to cubic-order terms in $\Delta\dot{V}^{(k)}$. From Eqs. 16 and 17 we obtain

$$\frac{\partial \phi}{\partial Z} = -\frac{1}{\Delta(\phi)}\left(\frac{\phi\dot{V}}{1 - Z} + \frac{\Delta\rho}{\rho_l}\frac{\phi}{\lambda} - \frac{\psi}{1 - Z}\right)$$
$$\frac{\partial \psi}{\partial Z} = \frac{\phi[Z, T^{(k)}] - \phi[Z, T^{(k-1)}]}{T - T^{(k-1)}} \quad (39)$$

Here we have replaced $\partial\phi/\partial T$ with a backwards difference, a method also used by Howells and coworkers.[10] This method is satisfactory for the final filtration stages because $\phi(Z, T)$ is slowly varying here unlike the initial stages.[4]

These compression zone equations are then solved from $Z_I^{(k)}$ to Z^* where solids conservation is satisfied:

$$I_1(Z^*) = L\phi_0 \quad (40)$$

If $Z_c = Z^*$ then the estimate of $\dot{V}^{(k)}$ was correct; otherwise, select a new estimate for $\dot{V}^{(k)}$ and solve again.

If the cake never unsaturates we increment the time steps until $\dot{V} = 0$ to within a set tolerance. The $T = \infty$ state of the

Table 1. System Values and Functions Used in Centrifugal Filtration Calculations

Drum radius	$r_m = 1.0$ m	Initial load	$r_f(0) = 0.5$ m
Density of liquid	$\rho_l = 1.0$ g/cm^3	Density of solids	$\rho_s = 2.0$ g/cm^3
Initial solids volume fraction	$\phi_0 = 0.1$	Gel point	$\phi_g = 0.15$
Compressive yield stress	$p_y(\phi) = p_0[(\phi/\phi_g)^5 - 1]$	Scaled initial load	$L = 0.75$
Hydrodynamic resistance	$R(\phi) = R_0(1 - \phi)^{-5.5}$	Liquid viscosity	$\mu_f = 0.001$ kg m^{-1} s^{-1}

system is known a priori; otherwise, we increment time until $T^{(k)} = T_u$, where $\phi[Z_c, T^{(k)}] = \phi_{cap}$ and the capillary compression stage is complete.

Cake Draining

For $T > T_u$, the liquid surface moves inside the cake $[Z_f(T) < Z_c(T)]$. This creates a region of unsaturated solids that grows as liquid continues to drain from the cake. Within the unsaturated solids layer, there is no buoyant force on the solids, so the solids pressure gradient becomes

$$\frac{\partial p_s}{\partial r} = \rho_s \omega^2 r \phi \qquad r_f(t) > r \geq r_c(t_u) \tag{41}$$

or

$$\frac{\partial P_s}{\partial Z} = -\frac{\rho_s}{\rho_l}\phi \qquad Z_f(T) < Z \leq Z_c(T_u) \tag{42}$$

in scaled form. The total pressure is continuous throughout the system, so that at the saturation boundary $Z_f(T)$, the total pressure is

$$P_t(Z_f) = P_a + \frac{\rho_s}{\rho_l}\int_{Z_f}^{Z_c} \phi dZ = P_s(Z_f^-) + P_l(Z_f^-) \tag{43}$$

The liquid pressure just inside the boundary is

$$P_l(Z_f^-) = P_a - p_{cap}^{max}[\phi(Z_f, T)] \tag{44}$$

(Note that the capillary pressure is maximal because the liquid–air interface is advancing through the cake.)

Thus, from Eq. 43,

$$P_s(Z_f^-) = \frac{\rho_s}{\rho_l}\int_{Z_f}^{Z_c} \phi dZ + p_{cap}^{max}[\phi(Z_f)] \tag{45}$$

or

$$P_s(Z_f^-) = \left(1 + \frac{\Delta\rho}{\rho_l}\right)[L\phi_0 - I_1(Z_f)] + p_{cap}^{max}(\phi_f) \tag{46}$$

(Note that we have assumed complete displacement of the fluid from the solids during the unsaturation stage.)

We neglect in the solids force balance and in the fluid volume balance any residual fluid contained in capillary necks between the particles. Also note that for purely centrifugally driven draining the liquid–vapor front is a stable boundary. This is in contrast to pressure-driven flows where local irregularities create vapor "fingers" through the cake. Because the liquid is being driven by its own weight and the vapor is simply filling space vacated by the liquid, there is no mechanism for the creation of an unstable liquid–vapor boundary. This is not to say that local variations in solids concentration will not give rise to an uneven interface. Rather, the method of cake formation will tend to minimize these variations and the variations that do occur will be reduced by the compression of the cake.

At T_u we know that $P_y[\phi(Z_f, T_u)] = p_{cap}^{max}$ because $\phi(Z_f, T_u) = \phi_{cap}$. Cake compression occurs if $P_s(Z, T) = P_y[\phi(Z, T)]$ at any point Z. If, however, $P_s(Z, T) < P_y[\phi(Z, T)]$ then the solids are immobile because there is insufficient solids stress to compress the cake locally. We note from Eq. 45 that the solid pressure at Z_f^- is the sum of the maximum capillary stress, which is a weak function of ϕ, and the weight of unsaturated solids, which is linear in ϕ. The compressive yield stress is a strong function of ϕ (ϕ^5 in our model). As the liquid–air interface advances through the cake, $\phi(Z_f, T)$ increases. The solids pressure at Z_f increases more slowly than the compressive yield stress at the volume fraction $\phi(Z_f, T)$ (see Figure 4). Thus the solids network will never experience sufficient stress to cause consolidation, and thus the cake is everywhere immobile for $T > T_u$.

Cake draining dynamics

The position of the liquid surface is determined from conservation of liquid volume (Eq. 6):

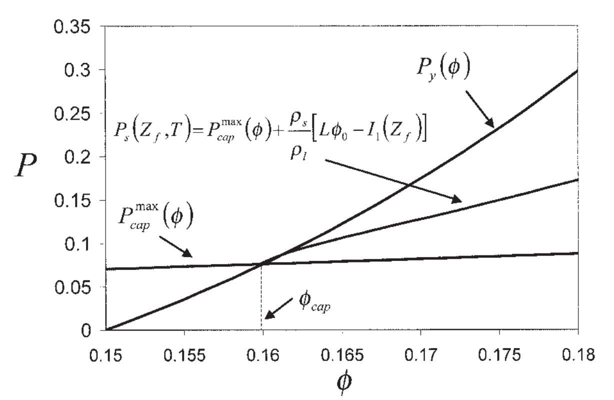

Figure 4. Compressive yield stress, maximum capillary pressure, and the solids pressure arising from capillary stress and weight of unsaturated solids during cake draining for P* = 5, M = 0.001, and F_{cap} = 2.0.

Note that as ϕ increases (corresponding to Z_f receding into the cake) the yield stress exceeds the solids pressure and thus the cake is immobile.

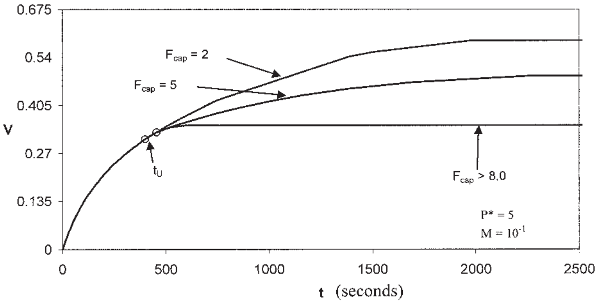

Figure 5. Volume expressed over time for P* = 5 and M = 0.1.

For $F_{cap} > 8.0$ the solids remain saturated. t_u is shown for plots of unsaturating cakes.

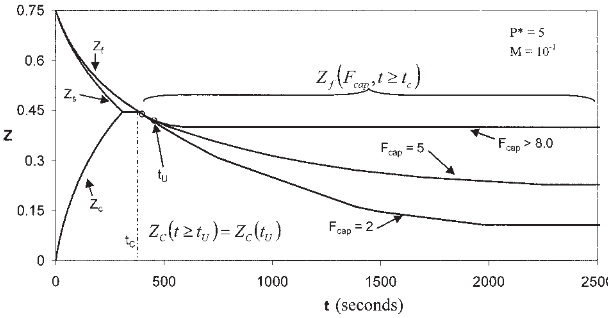

Figure 7. Boundary positions over time for P* = 5 and M = 0.1.

For $F_{cap} > 8.0$ the solids remain saturated. t_u is shown for plots of unsaturating cakes.

$$Z_f = L(1 - \phi_0) - V + I_1(Z_f) \tag{47}$$

and because $P_s(Z, T) < P_y[\phi(Z)]$ everywhere in the cake, we integrate Eq. 2 with $u = 0$ from $Z = 0$ to Z_f using Eqs. 8 and 44 to obtain

$$\dot{V} = \frac{Z_f - p_{cap}^{max}[\phi(Z_f, T)]}{\mu + I_2(Z_f)} \tag{48}$$

The immobile cake implies that $I_2(Z, T)$ is fixed at its $T = T_u$ form.

From Eq. 48 we can determine that the solids network will never fully unsaturate when a capillary stress is present, where full unsaturation would be indicated by $Z_f = 0$. From, Eq. 48 equilibrium ($\dot{V}$) is reached at the Z_f value at which

$$p_{cap}^{max}[\phi(Z_f, \infty)] = Z_f(\infty) \tag{49}$$

Cake draining algorithm

We first determine $Z_f(\infty)$ using the T_u solids concentration profile and Eq. 49. To calculate the time evolution of the cake draining stage, we systematically vary Z_f from $Z_f(T_u)$ to $Z_f(\infty)$. For each given value of $Z_f^{(k)}$, we calculate $V^{(k)}$, $\dot{V}^{(k)}$, and $T^{(k)}$ using the T_u solids profile information to evaluate Eqs. 47, 48, and 37, respectively.

Cake Compression and Draining Numerical Results

As discussed above, the total amount of liquid expressed during filtration is dependent on the capillary stress exerted on the liquid, so that expression stops when the capillary stress at the liquid surface is sufficient to reduce the liquid pressure at the membrane to zero. In Figures 5 and 6 the scaled volume expressed is plotted as a function of unscaled time ($R_0 = 4.486 \times 10^6$ kPa·s/m^2) for P* = 5 and 50, M = 0.1 and 10^{-3}, and various values of F_{cap}.

Boundary positions

The evolution of the regional boundaries in the cake and suspension over the filtration time are plotted in Figures 7 and 8 for P* = 5 and 50, M = 0.1 and 10^{-3}, and various values of F_{cap}. For $t > t_c$, Z_c has been omitted for clarity. Recall that for $t_c \leq t \leq t_u$, $Z_c = Z_f$ and for $t > t_u$, $Z_c(t) = Z_c(t_u)$.

Final solids concentration

The equilibrium solids concentration profiles at P* = 5 and 50, M = 0.1 and 10^{-3}, and various values of F_{cap} are plotted in Figures 9 and 10. This final concentration profile is reached at t_u for systems that unsaturate. Systems that remain saturated will limit to the final saturated profile as time goes to infinity and $\dot{V}$ goes to zero.

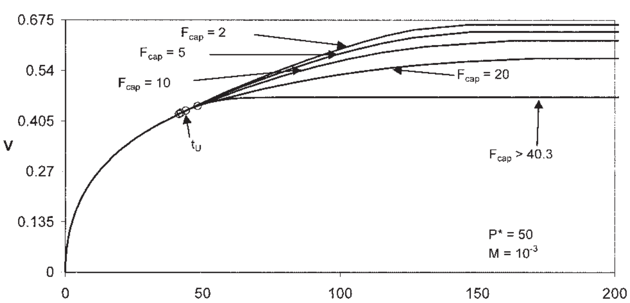

Figure 6. Volume expressed over time for P* = 50 and M = 0.001.

For $F_{cap} > 40.3$ the solids remain saturated. t_u is shown for plots of unsaturating cakes.

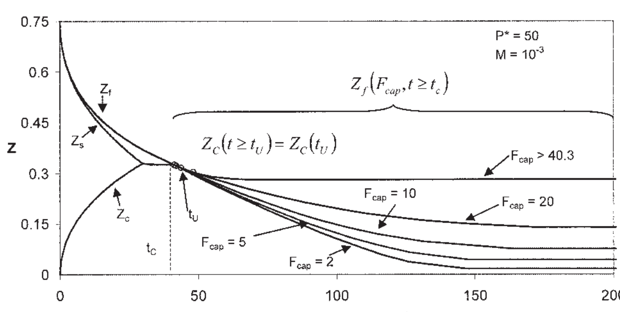

Figure 8. Boundary positions over time for P* = 50 and M = 0.001.

For $F_{cap} > 40.3$ the solids remain saturated. t_u is shown for plots of unsaturating cakes.

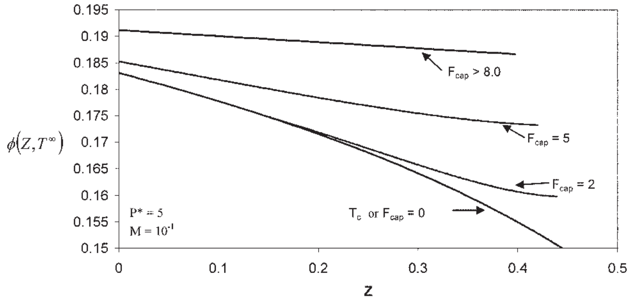

Figure 9. Final concentration profiles for P* = 5 and M = 0.1.

For $F_{cap} > 8.0$ the solids remain saturated.

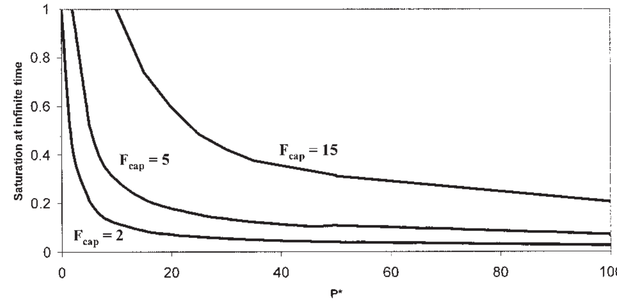

Figure 11. Cake saturation at infinite time for F_{cap} = 2, 5, and 15.

Note that the final saturation trends to unity as P* goes to zero. M dependency is negligible over the range $0.1 \geq \mathrm{M} \geq 0.0001$.

Cake saturation at infinite time

The final saturation of the filter cake is of interest for evaluating filter operation. Wakeman[1,2] uses fractional void space filled by liquid as a measure of the saturation of the cake. The quantity S_p, or $S_{p,\infty}$ at infinite time, is the volume of liquid remaining in the cake divided by the pore volume, or void space, in the cake. During cake compression and draining, S_p is easily calculated:

$$S_p = \frac{L(1 - \phi_0) - V}{Z_c - L\phi_0} \tag{50}$$

$S_{p,\infty} = 1$ for all cases where the cake remains saturated; otherwise, S_p will decrease as the cake drains, with the final saturation being a decreasing function of increasing pressure and decreasing capillary stress (see Figure 11).

Conclusions

The model investigated herein, combined with the cake formation algorithms shown earlier,[4] provides a complete description of the action of both the solid and liquid phases during an ideal batch centrifugal filtration operation from start to infinite time. The model requires knowledge of the rheological functions $p_y(\phi)$ and $R(\phi)$, along with the wetting properties of the suspension. In particular, the cake compression and draining portions of this model circumvent the use of pore characteristics as constituting the important property is the wetted area of solids.

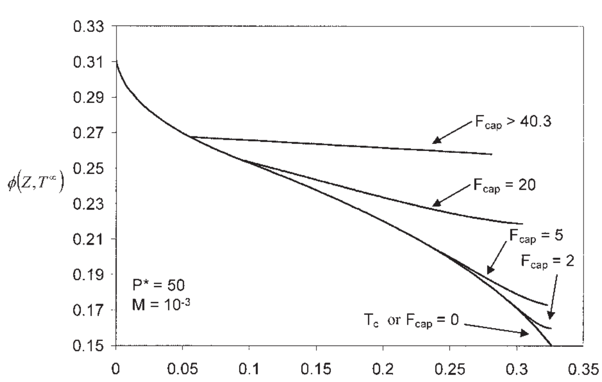

Figure 10. Final concentration profiles for P* = 50 and M = 0.001.

For $F_{cap} > 40.3$ the solids remain saturated.

The model presented here has been for an idealized, instantaneously, and uniformly loaded drum. The techniques used here should provide the basis to address more practical problems of optimal loading rates and drum spin-up.

Notation

$x^{(k)}$ = variable at time step k
x_∞ = variable at time = infinity
$\bar{A}_s$ = specific surface area of solids
D = diffusion coefficient
F_{cap} = capillary stress coefficient
$I_1(Z)$ = integral of the solids volume over the range 0–Z
$I_2(Z)$ = integral of the resistance to flow resulting from solids over the range 0–Z
$I_3(Z)$ = integral of the contribution to the rate of expression resulting from solids flux over the range 0–Z
L = scaled initial suspension volume (to be filtered) per unit length
M = membrane resistance divided by R_0
p_0 = prefactor for yield stress function
p_a = atmospheric pressure, inside and outside the drum
p_{cap} = capillary stress
p_l = liquid pressure
p_s = solids pressure
p_y = solids compressive yield stress
P* = pressure scaling factor divided by p_0
P_{cap} = scaled capillary stress
p_{cap}^{max} = maximum capillary stress at the local solids concentration
P_y = scaled solids compressive yield stress
q = scaled time derivative of solids concentration
r = radial coordinate
r_{eff} = average radius of curvature of the menisci at the solid–liquid–vapor interface
r_m = radius of the filter drum
$R(\phi)$ = hydrodynamic resistance
R_0 = hydrodynamic resistance at infinite dilution
R^* = resistance scaling factor
R_{mem} = membrane resistance
S_p = pore saturation: liquid volume inside the cake/pore volume of the cake
t = time
t^* = timescaling
t_c = time when the liquid surface intersects the cake surface
t_I = time at which the cake first immobilized at the membrane
t_u = time when the liquid surface moves inside the cake
T = scaled time
$\boldsymbol{u}$ = solids velocity
v = volume expressed per unit length
V = scaled volume expressed per unit length

$\dot{V}$ = scaled rate of volume expression per unit length
$\ddot{V}$ = scaled rate of change of rate of volume expression per unit length
Z = scaled radial coordinate

Greek letters

γ_{LV} = liquid–vapor surface tension
$\Delta(\phi)$ = scaled solids diffusion coefficient
ϕ = solids volume fraction
Φ = zero flux limit solids volume fraction
ϕ_0 = solids volume fraction of the feed also initial solids concentration
ϕ_{cap} = concentration where capillary stress equals the yield stress
ϕ_g = solids fraction at the gel point
ϕ_I = concentration at the immobilization boundary
$\phi \boldsymbol{u}$ = solids flux
$\lambda(\phi)$ = scaled hydrodynamic resistance
μ = scaled membrane resistance
μ_f = suspension viscosity
ψ = scaled solids flux
ρ_l = liquid density
ρ_s = solids density
$\Delta\rho$ = solids density minus liquid density
Θ_C = solid–liquid contact angle

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